

Operators for scattering of particles with spin

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Operators for simulating the scattering of two particles with spin are constructed. Three methods are shown to give the consistent lattice operators for PN , PV , VN and NN scattering, where P , V and N denote pseudoscalar meson, vector meson and nucleon. The projection method leads to one or several operators $O_{\Gamma,r,n}$ that transform according to a given irreducible representation Γ and row r . However, it gives little guidance on which continuum quantum numbers of total J , spin S , orbital momentum L or single-particle helicities $\lambda_{1,2}$ will be related with a given operator. This is remedied with the helicity and partial-wave methods. There first the operators with good continuum quantum numbers $(J,P,\lambda_{1,2})$ or (J,L,S) are constructed and then subduced to the irreps Γ of the discrete lattice group. The results indicate which linear combinations $O_{\Gamma,r,n}$ of various n have to be employed in the simulations in order to enhance couplings to the states with desired continuum quantum numbers. The total momentum of two hadrons is restricted to zero since parity P is a good quantum number in this case.

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1. Introduction

Most of hadrons, particularly the exotic ones, are resonances that appear as cross-section peaks in the strong scattering of lighter hadrons. This requires the study of two-hadron interactions by simulating two-hadron scattering on the lattice. The first step is to build the operators that create and annihilate the two-hadron system of the desired quantum numbers. Channels with two spinless hadrons have been extensively studied. The simulations of two-hadron systems where one or both hadrons carry non-zero spin focused mostly on partial wave $L = 0$. There is a great need for lattice results on further channels or higher partial waves in this case.

We construct the lattice operators for two-hadron scattering, where one or both hadrons carry spin and show that three independent methods lead to consistent results. The more detailed presentation [1] provides also the proofs of the methods, explicit expressions for operators and all necessary details to construct them. We consider channels that involve the nucleon N or/and vectors $V = J/\psi, \Upsilon_b, D^*, B^*, \dots$ which are (almost) stable under strong interactions. The PN or VN scattering is crucial for ab-initio study of baryon resonances and pentaquark candidates, PV is essential for mesonic resonances and tetraquark candidates, while NN is needed to grasp two-nucleon interaction and deuterium. The periodic boundary conditions in a box of size L are considered, where momenta p of non-interacting single hadrons are multiples of $2\pi/L$. We focus on the system with total momentum zero with the advantage that the parity is a good quantum number.

The resulting operators can be used to extract the discrete energies of eigenstates. These energies render the scattering phase shift via the well-known Lüscher relation [2] which originally considered two spin-less particles. This has been generalized to the scattering of two particles with spin by various authors, most generally by [3].

Certain aspects of constructing the lattice operators for scattering of particles with spin have already been considered [4, 5, 6, 7, 8, 5, 9, 3] before [1]. Despite all previous work, various aspects and proofs were lacking to build a reliable operator related to the desired continuum quantum numbers, for example partial-wave L or single-hadron helicities $\lambda_{1,2}$.

2. Single-hadron operators and their transformations

The single-hadron annihilation operators $H(p)$ need to have the following transformation properties under rotations R and inversion I in order to build two-hadron operators $H^{(1)}(p)H^{(2)}(-p)$ with desired transformation properties

$$RH_{m_s}(p)R^{-1} = \sum_{m'_s} D_{m_s m'_s}^s(R^{-1}) H_{m'_s}(Rp), \quad IH_{m_s}(p)I = (-1)^P H_{m_s}(-p). \quad (2.1)$$

For a particle at rest m_s is a good quantum number of the spin-component S_z . The m_s is generally not a good quantum number for $H_{m_s}(p \neq 0)$; in this case it denotes the eigenvalue of S_z for the corresponding field $H_{m_s}(0)$, which has good m_s . Our two-hadron fields are built from simple (non-

canonical) fields that satisfy (2.1), for example

$$P(p) = \sum_x \bar{q}(x) \gamma_5 q(x) e^{ipx} \quad (2.2)$$

$$V_{m_s=\pm 1}(p) = \frac{\mp V_x(p) + iV_y(p)}{\sqrt{2}}, \quad V_{m_s=0}(p) = V_z(p), \quad V_i(p) = \sum_x \bar{q}(x) \gamma_i q(x) e^{ipx}, \quad i = x, y, z$$

$$N_{m_s=1/2}(p) = \mathcal{N}_{\mu=1}(p), \quad \mathcal{N}_{m_s=-1/2}(p) = \mathcal{N}_{\mu=2}(p), \quad \mathcal{N}_{\mu}(p) = \sum_x \varepsilon_{abc} [q^{aT}(x) C \gamma_5 q^b(x)] q_{\mu}^c(x) e^{ipx}$$

where $\mathcal{N}_{1,2}$ are the upper two components of Dirac four-spinor $\mathcal{N}_{\mu=1,\dots,4}$ in the Dirac basis.

Another choice of building blocks could be the canonical hadron fields $H_{m_s}^{(c)}(p) \equiv L(p) H_{m_s}(0)$ obtained after a boost $L(p)$ from 0 to p . Those are less practical since they depend on the hadron mass m , energy E and velocity v , for example $N_{1/2}^{(c)}(p_x) \propto \mathcal{N}_1(0) + \frac{p_x}{m+E} \mathcal{N}_4$ and $V_{m_s=1}^{(c)}(p_x) = [-\gamma V_x(p_x) + iV_y(p_x)]/\sqrt{2}$ with $\gamma = (1 - v^2)^{1/2}$ [1].

3. Transformation properties of two-hadron operators

The two-hadron operators with zero momentum have good parity P and have to transform as

$$R O^{J,m_J}(P_{tot}=0) R^{-1} = \sum_{m'_J} D_{m_J m'_J}^J(R^{-1}) O^{J,m'_J}(0), \quad R \in O^{(2)}, \quad I O^{J,m_J}(0) I = (-1)^P O^{J,m_J}(0). \quad (3.1)$$

Such continuum-like operators will present an intermediate step below and the only relevant rotations will be those of the discrete group.

The continuum rotation group is reduced to the cubic group on a cubic lattice. It has 24 elements $R \in O$ for integer J and 48 elements $R \in O^2$ for half-integer J . The number of symmetry elements gets doubled to $\tilde{R} = \{R, IR\} \in O_h^{(2)}$ when inversion is a group element. The representations (3.1) with given J and m_J are reducible under $O_h^{(2)}$, so we seek the annihilation operators that transform according to the corresponding irreducible representation (irrep) Γ and row r

$$R O_{\Gamma,r} R^{-1} = \sum_{r'} T_{r,r'}^{\Gamma}(R^{-1}) O_{\Gamma,r'}, \quad R \in O^{(2)}, \quad I O_{\Gamma,r} I = (-1)^P O_{\Gamma,r}. \quad (3.2)$$

The systems with integer J transform according to irreps $\Gamma = A_{1,2}^{\pm}, E^{\pm}, T_{1,2}^{\pm}$, while systems with half-integer J according to $\Gamma = G_{1,2}^{\pm}, H^{\pm}$. We employ the same conventions for rows in all irreps as in [10], where explicit representations $T_{r,r'}^{\Gamma}(R)$ are given.

4. Two-hadron operators in three methods

Here we present two-hadron operators that transform according to (3.2) derived with three methods. The continuum-like operators (3.1) will appear as an intermediate step in two of the methods. Their correct transformation properties (3.1,3.2) are proven in Appendix of [1].

4.1 Projection method

A projector to the desired irrep Γ and row r on an arbitrary operator renders [11]

$$O_{|p|,\Gamma,r,n} = \sum_{\tilde{R} \in O_h^{(2)}} T_{r,r}^{\Gamma}(\tilde{R}) \tilde{R} H^{(1),a}(p) H^{(2),a}(-p) \tilde{R}^{-1}, \quad n = 1, \dots, n_{max}. \quad (4.1)$$

The $\tilde{R} \in O_h^{(2)}$ in (4.1) is the operator symbolizing rotations R possibly combined with inversions (R is reserved for rotations only) where the action of rotation R or inversion I on all $H = P, V, N$ is given in (2.1). The representation matrices $T^\Gamma(\tilde{R})$ for all elements \tilde{R} are listed for all irreps in Appendix A of [10].

The H^a are arbitrary single hadron lattice operators of desired $|p|$ and any p and m_s , for example operators (2.2) or their linear combinations. We have taken H^a with all possible combinations of direction p and polarizations of both particles m_{s1} and m_{s2} (for vectors we chose V_x, V_y and V_z as H^a). For fixed $|p|$, Γ and r one can get one or more linearly independent operators $O_{|p|,\Gamma,r,n}$ which are indicated indexed by n . As an illustration we present resulting PV operators for irrep T_1^+ and $|p| = 1$,

$$O_{|p|=1, T_1^+, r=3, n=1} \propto \sum_{p=\pm e_z} P(p) V_z(-p), \quad O_{|p|=1, T_1^+, r=3, n=2} \propto \sum_{p=\pm e_x, \pm e_y} P(p) V_z(-p), \quad n_{\max} = 2,$$

while others are listed in [1].

The projection method is very general, but it does not offer physics intuition what $O_{|p|,\Gamma,r,n}$ with different n represent in terms of the continuum quantum numbers. This will be remedied with the next two methods, that indicate which linear combinations of O_n correspond to certain partial waves or helicity quantum numbers.

4.2 Helicity method

The helicity λ is the eigenvalue of the helicity operator $h \equiv S \cdot p / |p|$. It is a good quantum number for a moving particle with any p (while m_s is a good quantum for $p = 0$ and $p \propto e_z$, but not in general). To obtain a helicity single-hadron annihilation operator $H_\lambda^h(p)$, one starts from an operator $H_{m_s=\lambda}(p_z)$ with momentum $p_z \propto e_z$ that has good m_s . This state is rotated from p_z to desired direction of p with R_0^p [12, 7]

$$H_\lambda^h(p) \equiv R_0^p H_{m_s=\lambda}(p_z) (R_0^p)^{-1}, \quad p_z \propto e_z, \quad |p_z| = p. \quad (4.2)$$

The upper index h indicates that the polarization index (λ) stands for the helicity of the particle (not m_s). Simple examples of $H_{p_z, m_s=\lambda}$ can be read-off from (2.2) and the action of R_0^p on them is given by (2.1). The arbitrary rotation R rotates the p and S in the same way, so the helicity $RH_\lambda^h(p)R^{-1} \propto H_\lambda^h(Rp)$ does not change [12, 7].

The two-hadron helicity operator is built from single-hadrons of given helicities $\lambda_{1,2}$ and arbitrary back-to-back momenta p within a given $|p|$

$$O_{|p|, J, m_J, \lambda_1, \lambda_2, \lambda} = \sum_{R \in O(2)} D_{m_J, \lambda}^J(R) R H_{\lambda_1}^{(1), h}(p) H_{\lambda_2}^{(2), h}(-p) R^{-1}. \quad (4.3)$$

The correct transformation properties (3.1) of the above operator are proven in Appendix of [1]. The final operator with desired parity $P = \pm 1$ is obtained by parity projection $\frac{1}{2}(\mathcal{O} + PI\mathcal{O}I)$

$$\begin{aligned} O_{|p|, J, m_J, P, \lambda_1, \lambda_2, \lambda} &= \frac{1}{2} \sum_{R \in O(2)} D_{m_J, \lambda}^J(R) R R_0^p [H_{m_{s1}=\lambda_1}^{(1)}(p_z) H_{m_{s2}=-\lambda_2}^{(2)}(-p_z) \\ &\quad + P I H_{m_{s1}=\lambda_1}^{(1)}(p_z) H_{m_{s2}=-\lambda_2}^{(2)}(-p_z) I] (R_0^p)^{-1} R^{-1}, \end{aligned} \quad (4.4)$$

where we have expressed H^h (4.2) with fields $H_{m_s}(p_z)$ (2.2) that have good quantum number m_s . The actions of inversion I and the rotation R on the fields H_{m_s} are given in (2.1). One chooses particular p for a fixed $|p|$ and performs rotation R_0^p from p_z to p . There are several possible choices of R_0^p , but they lead only to different overall phases for the whole operator (4.4) [1], which is irrelevant.

As an illustration we present PV operators in the $J^P = 1^+$ channel

$$O|p|=1, J=1, m_J=0, P=+, \lambda_V=0 \propto \sum_{p=\pm e_z} P(p) V_z(-p), \quad O|p|=1, J=1, m_J=0, P=+, \lambda_V=1 \propto \sum_{p=\pm e_x, \pm e_y} P(p) V_z(-p),$$

where the simplest choice $p = p_z = (0, 0, 1)$ and $R_0^p = \mathbf{1}$ in (4.4) can be used.

The helicity operators (4.4) would correspond to irreducible representations only for the continuum rotation group. These represent reducible representation under the discrete group $O^{(2)}$. In simulations it is convenient to employ operators, which transform according to irreducible representations Γ and row r of $G = O^{(2)}$. Those are obtained from O^{J, m_J} by the subduction [13, 14]

$$O_{|p|, \Gamma, r}^{[J, P, \lambda_1, \lambda_2, \lambda]} = \sum_{m_J} \mathcal{S}_{\Gamma, r}^{J, m_J} O_{|p|, J, m_J, P, \lambda_1, \lambda_2, \lambda}. \quad (4.5)$$

The subduction coefficients \mathcal{S} are given in Appendices of [13, 14] for all irreps¹. We expect that the subduced operators $O_{|p|, \Gamma, r}^{[J, P, \lambda_1, \lambda_2, \lambda]}$ will carry the memory of continuum $J, \lambda_1, \lambda_2, \lambda$ and will dominantly couple to eigenstates with these quantum numbers [13].

4.3 Partial-wave method

Often one is interested in the scattering of two hadrons in a given partial wave L . The orbital angular momentum L and total spin S are not separately conserved, so several (L, S) combinations can render the same J, m_J and P , which are good quantum numbers. Nevertheless, the L and S are valuable physics quantities to label continuum annihilation field operator:

$$O|p|, J, m_J, S, L = \sum_{m_L, m_S, m_{s1}, m_{s2}} C_{L m_L, S m_S}^{J m_J} C_{s1 m_{s1}, s2 m_{s2}}^{S m_S} \sum_{R \in O} Y_{L m_L}^*(\widehat{R}p) H_{m_{s1}}^{(1)}(Rp) H_{m_{s2}}^{(2)}(-Rp). \quad (4.6)$$

The operator has parity $P = P_1 P_2 (-1)^L$ and its correct transformation property under rotation (3.1) is demonstrated in Appendix of [1]. The operator was considered for nucleon-nucleon scattering in [4] (this reference uses $Y_{L m_L}$ where we have $Y_{L m_L}^*$). The C are Clebsch-Gordan coefficients, p is an arbitrary momentum with desired $|p|$, $Y_{L m_L}$ is a spherical harmonic. Simple choices of one-particle operators H are listed in (2.2). Here is an example of PV operators with the same J and S

$$O|p|=1, J=1, m_J=0, L=0, S=1 \propto \sum_{p=\pm e_x, \pm e_y, \pm e_z} P(p) V_z(-p),$$

$$O|p|=1, J=1, m_J=0, L=2, S=1 \propto \sum_{p=\pm e_x, \pm e_y} P(p) V_z(-p) - 2 \sum_{p=\pm e_z} P(p) V_z(-p).$$

The operators that transform according to irrep Γ and row r of $G = O^{(2)}$ are obtained from O^{J, m_J} by the subduction [13] using the same coefficients \mathcal{S} as in (4.5)

$$O_{|p|, \Gamma, r}^{[J, S, L]} = \sum_{m_J} \mathcal{S}_{\Gamma, r}^{J, m_J} O_{|p|, J, m_J, S, L}. \quad (4.7)$$

¹For the T_1 we choose rows as x, y, z which differs from [13, 14], and those \mathcal{S} are listed in Appendix of [1].

One expects that the subduced operators $O_{|p|,\Gamma,r}^{[J,S,L]}$ carry the memory of continuum J, S, L and dominantly couple to eigenstates with these quantum numbers.

5. Operators for PV, PN, VN and NN scattering

The explicit expressions for operators $H^{(1)}(p)H^{(2)}(-p)$ for PV, PN, VN and NN scattering in three methods are collected in our longer publication [1]. Operators for the lowest two momenta $|p| = 0, 1$ (in units of $2\pi/L$) are presented for all irreducible representations. Expressions for higher $|p|$ can be obtained using the general expressions (4.1), (4.4,4.5) and (4.6,4.7) for the three methods, respectively.

Let us illustrate the results on operators for PV scattering in the irreducible representation T_1^+ . This irrep contains states with positive parity and $J = 1$ (as well as $J \geq 3$). The operators for the row $r = 3$ (z -component) are

$$|p| = 0 : \quad O_{T_1^+} = O_{T_1^+}^{[J=1,L=0,S=1]} = P(0)V_z(0) \quad (5.1)$$

$$\begin{aligned} |p| = 1 : \quad & O_{T_1^+,n=1} = P(e_z)V_z(-e_z) + P(-e_z)V_z(e_z) \\ & O_{T_1^+,n=2} = P(e_x)V_z(-e_x) + P(-e_x)V_z(e_x) + P(e_y)V_z(-e_y) + P(-e_y)V_z(e_y) \\ & O_{T_1^+}^{[J=1,P=+,\lambda_V=\pm 1,\lambda_P=0]} = O_{T_1^+,n=2} \\ & O_{T_1^+}^{[J=1,P=+,\lambda_V=0,\lambda_P=0]} = O_{T_1^+,n=1} \\ & O_{T_1^+}^{[J=1,L=0,S=1]} = O_{T_1^+,r=1,n=1} + O_{T_1^+,n=2} \\ & O_{T_1^+}^{[J=1,L=2,S=1]} = -2 O_{T_1^+,r=1,n=1} + O_{T_1^+,n=2} . \end{aligned} \quad (5.2)$$

The projection method renders two linearly independent interpolators $O_{T_1^+,n=1,2}$ at $|p| = 1$ for each row. This method does not tell which partial-waves and single-hadron helicities correspond to each operator. This is remedied by the partial-wave and helicity² operators. The expressions $O_{T_1^+}^{[J=1,L,S=1]}$ (5.1) indicate which linear combinations of $O_{T_1^+,n}$ need to be employed to study $L = 0$ or $L = 2$ partial waves. Note that both partial waves inevitably contribute to the same $J^P = 1^+$ channel even in the continuum PV scattering with $S = 1$. The $O_{T_1^+}^{[J=1,P=+,\lambda_V,\lambda_P]}$ (5.1) indicate that $O_{T_1^+,n=1}$ is relevant for $\lambda_V = 0$, while $O_{T_1^+,n=2}$ is relevant for $|\lambda_V| = 1$. All methods lead to two linearly independent operators that are consistent with each other.

In general, one or several linearly independent operators $O_{\Gamma,r,n}$ arise from the projection method for PV, PN, VN and NN scattering in given irrep Γ . The explicit partial-wave operators in [1] indicate which linear combinations of $O_{\Gamma,r,n}$ are relevant to study the channel (J, L, S) . The expressions for helicity operators in [1] tell us which linear combinations of $O_{\Gamma,r,n}$ are relevant to study scattering with given $(J, P, \lambda_1, \lambda_2)$. All three methods render the same number of linearly independent operators, which also agrees with the number based on [9]. The explicit expressions for operators with $|p| = 0, 1$ also show that the three methods lead to the consistent results, i.e. operators from partial-wave or projection method can always be expressed as a linear combination of operators from projection method.

²We consider helicity only for the case $|p| \neq 0$.

6. Conclusions

We construct two-hadron interpolators which are relevant to simulate PV , PN , VN or NN scattering using quantum field theory on the lattice. Here P , V and N denote pseudoscalar, vector and nucleon, respectively. The focus is on the case with total-momentum zero where parity is a good quantum number. The projection method is a general mathematical tool which leads to one or several operators $O_{\Gamma,r,n}$ that transform according to given irrep Γ and row r , but it does not give much insight on the underlying continuum quantum numbers. The partial-wave and the helicity methods indicate which linear combinations $O_{\Gamma,r,n}$ of various n have to be employed in the in order to enhance couplings to the states with desired continuum quantum numbers. The partial-wave method renders operators $O_{\Gamma,r}^{[J,S,L]}$ with enhanced couplings to two-hadron states in partial wave L , total spin S and total angular momentum J . The helicity method provides operators $O_{\Gamma,r}^{[J,P,\lambda_1,\lambda_2]}$ where each hadron has good helicity $\lambda_{1,2}$. All three formally independent methods lead to consistent results.

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